DOCUMENT RESUME

ED 076 404 SE 015 952

AUTHOR	Boyer, Lee E.; And Others
TITLE	Teaching Fractions with the Number Line. Mathematics Series No. 2.
INSTITUTION	Pennsylvania State Dept. of Public Instruction, Harrisburg.
PUB DATE	62
NOTE	14p.
EDRS PRICE	MF-$0.65 HC-$3.29
DESCRIPTORS	Curriculum; *Elementary School Mathematics; *Fractions; *Instruction; Mathematics Education; Number Concepts; *Teaching Guides
IDENTIFIERS	*Number Line

ABSTRACT

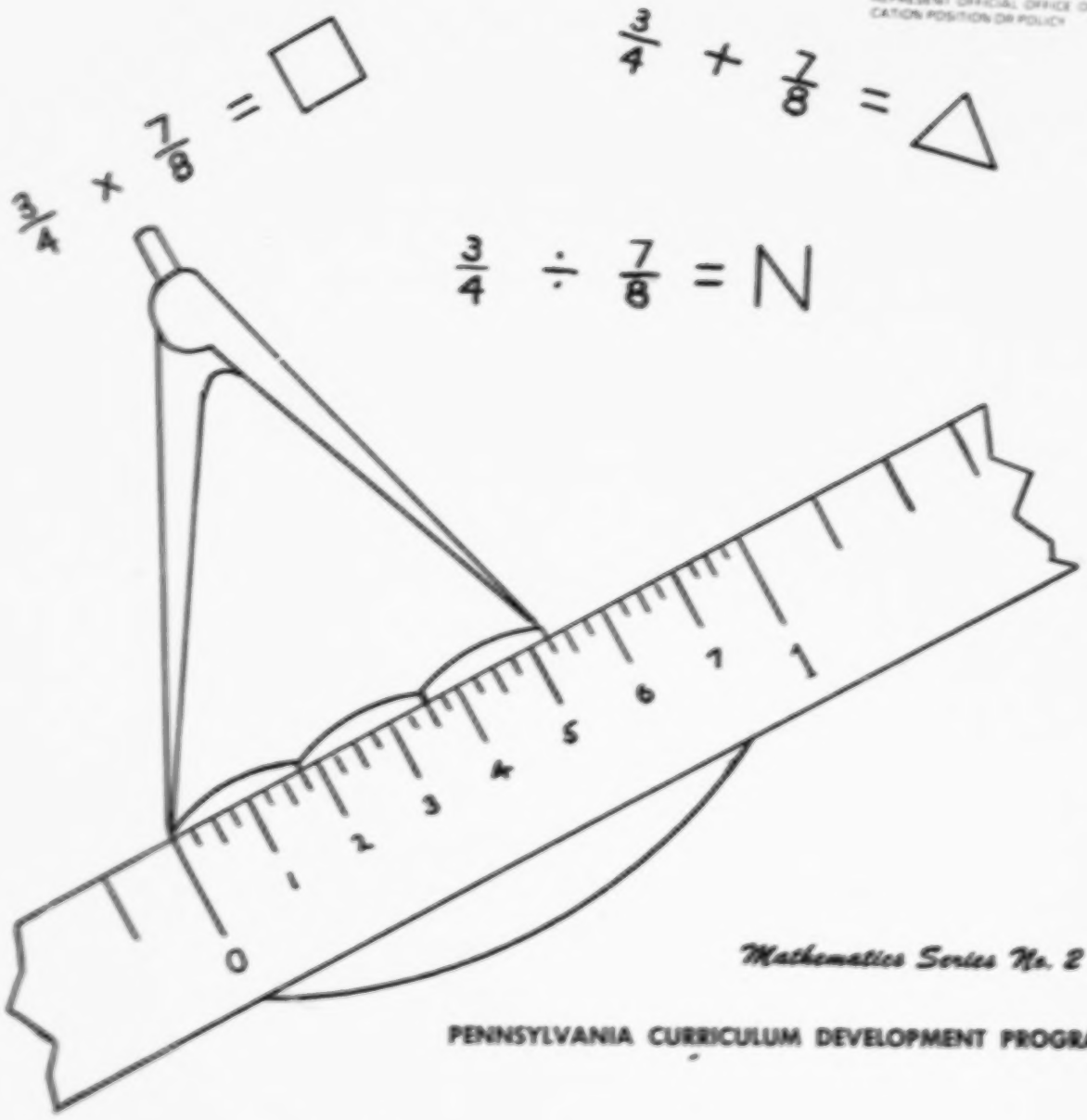
This booklet explains in detail how a number line can be used to demonstrate adding, subtracting, multiplying, and dividing fractions. The fractions 5/7 and 2/3 are first used as examples for each operation, then results are generalized to operations on a/b and c/d. (Related documents are SE 015 950 and SE 015 951.) (DT)

FILMED FROM BEST AVAILABLE COPY

TEACHING FRACTIONS WITH THE NUMBER LINE

ED 076404

U.S. DEPARTMENT OF HEALTH
EDUCATION & WELFARE
OFFICE OF EDUCATION
THIS DOCUMENT HAS BEEN REPRODUCED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT. POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL OFFICE OF EDUCATION POSITION OR POLICY



Mathematics Series No. 2

PENNSYLVANIA CURRICULUM DEVELOPMENT PROGRAM

SE 015 952

Commonwealth of Pennsylvania
DEPARTMENT OF PUBLIC INSTRUCTION

Harrisburg
1962

Reprint—1965

COMMONWEALTH OF PENNSYLVANIA

WILLIAM W. SCRANTON

Governor

GEORGE W. HOFFMAN

Acting Superintendent of Public Instruction

STATE BOARD OF EDUCATION

Dr. Otis C. McCreery
Chairman

Council of Basic Education

Mrs. Albert M. Greenfield
Chairman

Joseph F. Burke
Dr. Paul S. Christman
Parke H. Lutz
Mrs. Emaline K. Mohr
Dr. William M. Potter
Walter F. Wilmarth

Council of Higher Education

Charles G. Simpson
Chairman

Frank N. Hawkins
Dr. Katharine E. McBride
Dr. Otis C. McCreery
Gail L. Rose
James H. Rowland, Esq.
Dr. Leonard N. Wolf

Members at Large

Mario C. Celli
Ira C. Gross
Duane E. Wilder

Severino Stefanon
Secretary to the Board

First printing—10M—March, 1962
Second printing—10M—September, 1963
Third printing—10M—May, 1965

MESSAGE FROM THE SUPERINTENDENT OF PUBLIC INSTRUCTION

"The moving finger writes; and, having writ, moves on," noting changes which are occurring inexorably in our environment. Our technology changes in an ever more revolutionary fashion, and we must so educate our youth that they can make continual adjustments. In school mathematics this requires more emphasis on structure and generalizations to furnish a broad background which a capable individual can use in making adaptations.

Appropriate changes in emphasis in the secondary school mathematics program necessitate both alteration and acceleration in the program of the elementary schools. Experimental programs have shown that much more mathematics can be taught to younger children than had previously been thought possible. The arithmetic program can be taught more efficiently; algebraic and geometric concepts can be introduced effectively as early as the primary grades; and modern approaches can be used to provide both motivation and understanding. Another area of great importance is the combined approach to science and mathematics in the elementary grades. Steps are being taken to effect a unified approach along these lines in our Commonwealth. This combination makes it absolutely necessary to introduce in the elementary grades many mathematical concepts formerly developed in classes in mathematics on a more advanced level.

For the most part, we do not wish to make wholesale, sweeping changes. Rather we hope to develop materials which will enable the teacher to make transitions gradually, hoping in this way to build a strong foundation for further improvements.

George W. Hoffman

Acting Superintendent of Public Instruction

PREFACE

Teachers have little difficulty demonstrating the facts of addition, subtraction, multiplication, and division when whole numbers, or integers, are involved. A glance at both hands suffices to indicate the reasonableness of $4 + 1 = 5$; or that two 5's are 10. The use of the hands can be supplemented by using pencils, splints of wood, or cut-out stars, apples, and ducks in connection with the flannel board to provide ample illustrations of the reasonableness of the number facts involving the four fundamental operations as long as the work pertains to integers only.

Even if one fraction is combined with an integer the operations are still fairly easily demonstrated with concrete objects such as circular pieces of cardboard on which sectors are marked, measuring devices on which the subdivisions are marked, or patterned materials such as the square-blocked Hershey milk chocolate bar.

However, when two fractions are involved number fact demonstrations with concrete objects are much more difficult. To bridge this difficulty the School Mathematics Study Group[1] suggests using a "number line" technique. This technique was mentioned on page 59 of the September 22, 1961 issue

[1]The School Mathematics Study Group is a group of college mathematicians and teachers of school mathematics working under the direction of Professor Edward G. Begle, formerly at Yale University, now at Stanford University. This Study Group is supported financially by the National Science Foundation. Its purpose is to create teachable mathematics courses.

of Time magazine.[2] Since the Study Group's idea seems educationally sound and Time's description too sketchy for classroom use, this booklet will explain in detail how a number line can be used to demonstrate the reasonableness of the four fundamental operations of arithmetic when applied to fractions.

---

[2]The chief aim (of the SMSG materials) is a firm grasp of "real numbers," which form the central number system of mathematics. Real numbers include rational numbers, meaning whole numbers that can be expressed exactly as ratios or fractions. To visualize this structure, Begle presents a ruler-like "number line"; rational numbers are fractional lengths along the line. They have a beautiful unity that can be shown by dividing a segment of the number line into four equal parts. When the student subdivides three of the four parts into thirds, he discovers that two of these thirds make exactly one-half of the full unit. He thus "sees" fractional multiplication, which he can use to check the fact, quite abstruse to math beginners, that 2/3 times 3/4 equals 1/2.

TEACHING FRACTIONS WITH THE NUMBER LINE

Introduction. A "number line" is an indicated direction on which numbers are located sequentially and in such a manner that each pair of successive whole numbers is the same unit distance apart. For example, a foot rule, a thermometer scale, and the equator are illustrative of the properties of a number line. In mathematics we think of a number line as a non-ending straight line each point of which corresponds to one and only one of the real numbers.[3] The use of a number line in arithmetic is significant since it constitutes a connecting link between arithmetic and advanced mathematics. All counting in arithmetic can be associated with the number line concept. All graphing in algebra, trigonometry, analytic geometry, and calculus uses a number line as the basic line of reference.

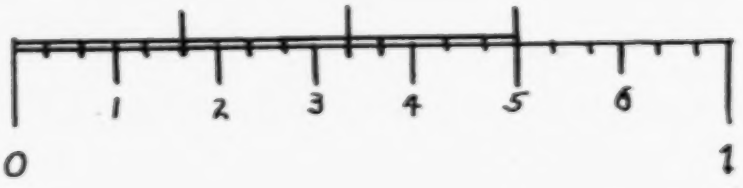
The concept of a number line is not difficult to comprehend. When the child stretches a tape line or applies a yardstick he is using the number line idea; yet, a number line is thought of as extending infinitely far in both directions and containing infinitely many points. This duality of simplicity and complexity corresponds, of course, to the simplicity and complexity of our real numbers whose use can challenge our most ambitious boys and girls.

[3]The set of real numbers includes zero and all the positive and negative integers, fractions, and irrational numbers. It does not include the imaginary numbers.

Multiplication. To multiply $\frac{5}{7}$ by $\frac{2}{3}$ is equivalent to asking, how much is $\frac{2}{3}$ of $\frac{5}{7}$? To determine the answer to this query by means of a number line technique we use the segment of a number line extending from zero to one. Thus,

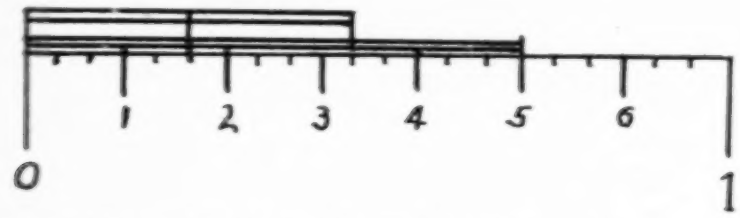
We, then, divide the line into seven parts[4] and indicate our concern with five of these seven parts by broadening the line segment extending over them. Later, we divide each of the seven parts into thirds. Thus,

The smallest divisions divide the line from 0 to 1 into twenty-one parts. The double line representing five sevenths of 1, or simply five sevenths contains fifteen of the small parts. We then divide the length covered by the fifteen small parts into thirds, counting by fives from zero, by placing three vertical lines above the double line. Thus,



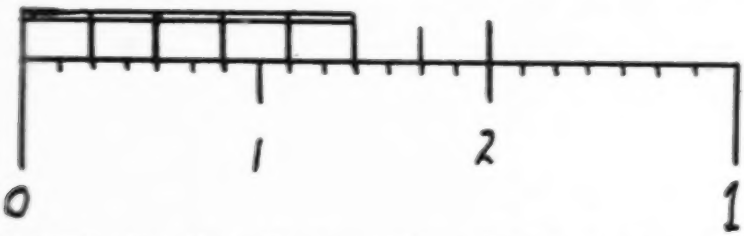
[4]All of the line divisions suggested in this booklet are constructed free-hand and thus are only approximately equal divisions. Logically equal divisions can be determined by using the Euclidean geometry construction method for "dividing a line into any number of equal parts."

Since we want to know how much <u>two</u> thirds of five sevenths is we will draw a double line over two of these thirds. Thus,



By counting on the number line the small parts under the double line it is obvious that <u>two</u> thirds of <u>five</u> sevenths is <u>ten</u> twenty-firsts of one or $\frac{10}{21}$. That is, $\frac{5}{7}$ multiplied by $\frac{2}{3}$ equals $\frac{10}{21}$.

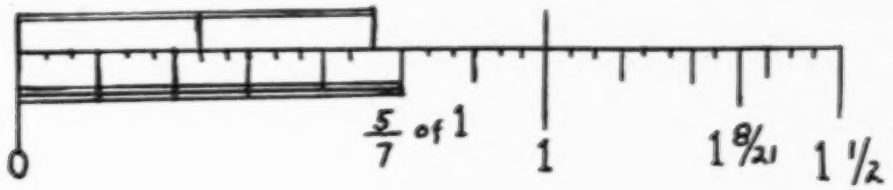
Since multiplication is commutative, we could have asked, how much is $\frac{5}{7}$ of $\frac{2}{3}$? The representative sketch would then have taken the following form



This sketch indicates that five sevenths of two thirds is again ten twenty-firsts of one or $\frac{10}{21}$. That is, $\frac{5}{7}$ of $\frac{2}{3}$ or $\frac{5}{7} \times \frac{2}{3} = \frac{10}{21}$.

<u>Addition</u>. To add $\frac{5}{7}$ and $\frac{2}{3}$ we use the segment of a number line extending from zero to one and one-half. Thus,

Again, we divide the segment from zero to one into sevenths and the sevenths into thirds. Thus,



Above the segment we indicate <u>two</u> thirds of 1, or simply two thirds, by a double line. Below the segment we indicate <u>five</u> sevenths of 1, or simply five-sevenths, by a triple line. To determine the sum of five-sevenths and two-thirds we count beyond the $\frac{5}{7}$ mark an additional fourteen small parts, the number of parts appearing under the double line which represents 2, to locate the point $1\ \frac{8}{21}$ of one or $1\ \frac{8}{21}$. That is,

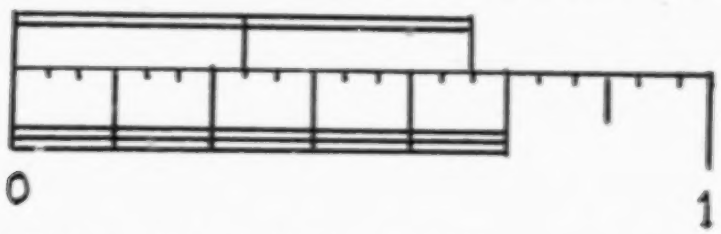
$\frac{5}{7} + \frac{2}{3} = \frac{15 + 14}{21} = \frac{29}{21} = 1\ \frac{8}{21}$.

<u>Subtraction</u>. The preceding sketch also illustrates that five sevenths less two thirds is simply one twenty-firsts of one or $\frac{1}{21}$. That is,

$\frac{5}{7} - \frac{2}{3} = \frac{15 - 14}{21} = \frac{1}{21}$.

<u>Division</u>. First, recall that division is essentially a record of repeated subtractions. For example, to divide twenty-two by five we determine the quotient by noting that <u>four</u> fives can be subtracted from twenty-two before we have a remainder less than the divisor, five. Thus, $22 \div 5 = 4$ with a remainder of 2. But four is only a partial quotient. The complete quotient must state the entire number of times the divisor, as a unit, is contained in the dividend. Therefore, the remainder two must be expressed as a <u>part of the divisor</u>, five. That makes the complete quotient four and two-fifths. Or, $22 \div 5 = 4\ \frac{2}{5}$.

To divide $\frac{5}{7}$ by $\frac{2}{3}$ using the number line technique we draw a segment of the number line extending from zero to one. Then divide the segment into sevenths and the sevenths into thirds as before. Below the segment we indicate the dividend, five sevenths, with a triple line. Above the line, we indicate, the divisor, two thirds, with a double line. Thus



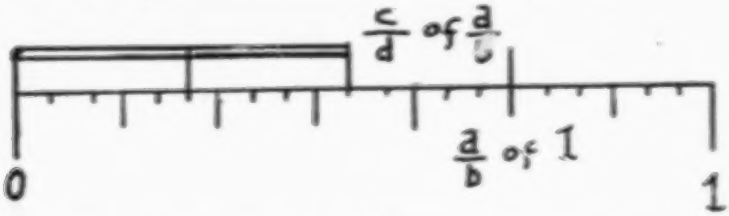
It is obvious that two-thirds, the divisor (represented by fourteen of the smallest parts), can be subtracted <u>once</u> from five-sevenths, the dividend (represented by fifteen of the smallest parts) and that the remainder - one small part - would constitute one fourteenth of the divisor. Thus, two-thirds can be subtracted from five-sevenths one and one fourteenth times (of) itself. That is, $\frac{5}{7} \div \frac{2}{3} = \frac{5}{7} \times \frac{3}{2} = \frac{15}{14} = 1\ \frac{1}{14}$.

It should be noticed that the fraction $\frac{15}{14}$, the ratio between the number of smallest parts in the dividend and the number of the smallest parts in the divisor, is another form of the complete quotient.

Having demonstrated the results of the four fundamental operations, with fractions in this manner, it is believed that pupils will be more appreciative of the shorter, but less meaningful, mechanical technique of performing the operations. Furthermore, if a question arises, the number line technique can always be used to demonstrate the correct result in a visual manner.

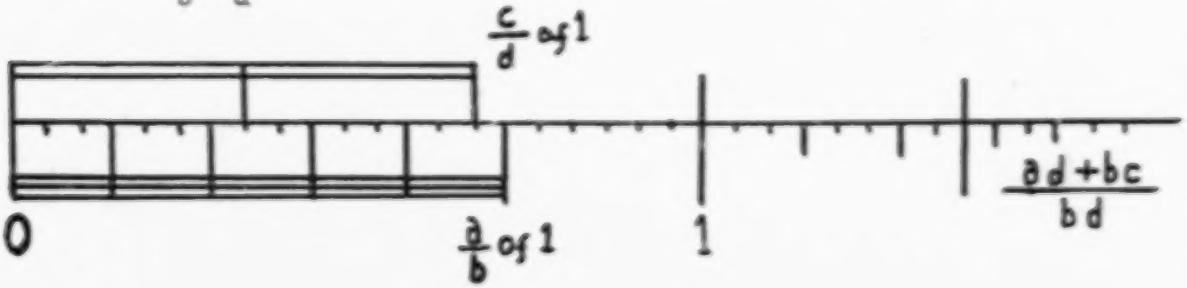
In all of the above operations a single pair of fractions was used. In the sketches below the general fractions $\frac{a}{b}$ and $\frac{c}{d}$ are considered. The line markings correspond to those used with the fractions $\frac{5}{7}$ and $\frac{2}{3}$ so that the reader can use his previous experience to interpret the generalized statements.

Multiplication. $\frac{a}{b} \times \frac{c}{d}$ is equivalent to finding $\frac{c}{d}$ of $\frac{a}{b}$.



By dividing the segment of length one into b parts and dividing each b into d smaller parts it is evident that the segment of length one is always divided into bd of the smallest parts. The heavy portion of the segment representing $\frac{a}{b}$ of one will always contain ad of the smallest parts. The procedure then divides the ad small parts into d parts above the line. The number of d parts will be $\frac{ad}{d}$ or a. The product of $\frac{a}{b} \times \frac{c}{d}$ is represented by c of these a parts. These will number c x a or ac. Since there are bd of the smallest parts in the segment of length one, the product of $\frac{a}{b}$ and $\frac{c}{d}$ will always be $\frac{ac}{bd}$ of one. That is, $\frac{a}{b} \times \frac{c}{d}$ will always equal $\frac{ac}{bd}$.

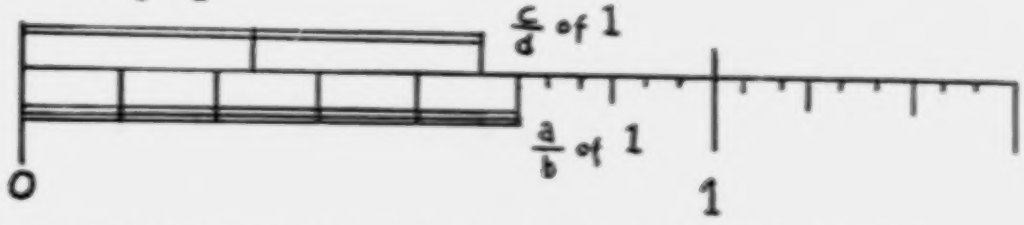
Addition. $\frac{a}{b} + \frac{c}{d}$



By dividing the segment of length one into b parts and dividing each b part into d smaller parts it is evident that the segment of length one is always divided into bd of the smallest parts. The fraction $\frac{a}{b}$ will always be represented by ad of the smallest parts. The fraction $\frac{c}{d}$ will always be represented by dividing the bd smallest parts by d getting $\frac{bd}{d}$ or b parts. But the fraction $\frac{c}{d}$ will always be represented by c of these b parts or bc of the smallest parts. The sum of the two fractions will then be $ad + bc$ of the smallest parts or $\frac{ad + bc}{bd}$ of one or simply $\frac{ad + bc}{bd}$. Thus, $\frac{a}{b} + \frac{c}{d}$ will equal $\frac{ad + bc}{bd}$.

Subtraction. $\frac{a}{b} - \frac{c}{d}$ (assuming that $\frac{c}{d}$ is less than $\frac{a}{b}$) is obviously $\frac{ad - bc}{bd}$ of one or $\frac{ad - bc}{bd}$.

Division. $\frac{a}{b} \div \frac{c}{d}$



Here again the segment of length one will be divided into bd of the smallest parts as before. The fraction $\frac{a}{b}$ will always be represented by ad of the smallest parts. As before, $\frac{c}{d}$ will always be represented by bc of the smallest parts. The quotient is therefore $\frac{ad}{bc}$. That is, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$.

Bibliography. Three references are suggested which provide an introduction to the number line and its use:

1. Brumfiel, Charles; Eicholz, Robert; Shanks, Merrill, Introduction to Mathematics. Addison-Wesley, Reading, Mass., 1961. pp. 128-135, 168-172.

2. Page, David A., Number Lines, Functions, and Fundamental Topics. University of Illinois Arithmetic Project, Urbana, Ill. 1961.

3. Twenty-fifth Yearbook, Instruction in Arithmetic. National Council of Teachers of Mathematics, Washington, D.C. 1961. pp. 55-58.

ACKNOWLEDGMENTS

The Department appreciates the efforts of Dr. Lee E. Boyer, Coordinator, NDEA In-Service Programs, in the preparation of the manuscript for this publication.

* * * * * * * * * * * *

Department Staff assisting:

Carl E. Heilman, Mathematics Specialist, NDEA

Thomas N. McCreary, Mathematics Specialist, NDEA

* * * * * * * * * * * *

MATHEMATICS SERIES

1. Properties of Numeration Systems
2. Teaching Fractions with the Number Line
3. Introduction to Sets and Set Notation
4. Numeration Systems with Bases Other Than Ten
5. Introducing Number Lines
6. Teaching Combinations with the Abacus